

VALIDITY OF THE GENERALIZED SECOND LAW OF THERMODYNAMICS OF THE UNIVERSE BOUNDED BY THE EVENT HORIZON IN BRANE SCENARIO

Nairwita Mazumder*, Subenoy Chakraborty†.

¹*Department of Mathematics, Jadavpur University, Kolkata-32, India.*

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In this paper, we examine the validity of the generalized second law of thermodynamics of the universe bounded by the event horizon in brane-world gravity. Here we consider homogeneous and isotropic model of the universe in one case where it is filled with perfect fluid and in another case holographic dark energy model of the universe has been considered.

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I. INTRODUCTION

The discovery of Hawking radiation of black holes in the semi-classical description, put black hole to behave as a black body and there is emission of thermal radiation. The temperature of the black hole (known as Hawking temperature) is proportional to the surface gravity at the horizon while the entropy of the black hole is related to the area of the horizon i.e. the above thermodynamic quantities are related to the geometry of space-time which is characterized by the Einstein field equations. So naturally one may speculate some relationship between black hole thermodynamics and Einstein equations. It is Jacobson [1] who first showed that it is possible to derive Einstein field equations from the first law of thermodynamics for local Rindler causal horizons. Subsequently, Padmanabhan [2] derived the first law of thermodynamics on the horizon, starting from Einstein equations for general static spherically symmetric space time. Recently, to investigate the profound connection between gravity and thermodynamics, Bamba et al [3] have examined the equivalence of modified gravity equation to the clausius relation.

Subsequently, this nice equivalence between black hole thermodynamics and the Einstein gravity raises the challenging issue whether there is any relationship between the thermodynamics of space-time and the nature of gravity. In fact if we assume the universe as a thermodynamical system bounded by the apparent horizon, its temperature and entropy are given by $T_A = \frac{1}{2\pi R_A}$, $S_A = \frac{\pi R_A^2}{G}$ and the first law of thermodynamics is equivalent with Friedmann equations on the apparent horizon. Also this equivalence still remains in higher dimensional gravity theories i.e. f(R) gravity, Lovelock gravity etc. In the context of brane world scenario, Ge [4] derived the Friedmann-like equations for the RS-II model using the first law of thermodynamics and Israel's junction conditions on the apparent horizon.

So far, the thermodynamical analysis of the universe is mainly restricted to the apparent horizon. In standard big bang cosmology event horizon does not exist. However, the cosmological event horizon can be distinguished from the apparent horizon in a general accelerating universe dominated by dark energy with an equation of state $\omega_D \neq -1$. Further, using the usual definition of temperature and entropy as in the apparent horizon Wang et al [5] have shown that both first and second law of thermodynamics break down on the cosmological event horizon. They have argued that the event horizon is a global feature of space-time while first law is valid to nearby states of local thermodynamic equilibrium. Also it is speculated that in the non-static universe, thermodynamical quantities on the cosmological event horizon may not be as simple as in the static space-time. Moreover, one may say the region within the apparent horizon as the Bekenstein system because Bekenstein's entropy mass bound ($S \leq 2\pi R_E$) and entropy-area bound ($S \leq \frac{A}{4}$) are satisfied in this region. As Bekenstein bounds are universal and all gravitationally stable special regions with weak self-gravity should satisfy Bekenstein bounds so the corresponding thermodynamical system is termed as a Bekenstein system.

* nairwita15@gmail.com

† schakraborty@math.jdvu.ac.in

In the context of recent observational evidences, the event horizon exists due to accelerating phase of the universe. As the universe bounded by the event horizon is not a Bekenstein system so Bekenstein's entropy-area relation and also Hawking temperature may not hold in the event horizon. Hence, from thermodynamical point of view, it is interesting to determine the temperature and entropy on the event horizon so that first law of thermodynamics is always true there. As a first step to have some inside we examine the validity of the generalized second law of thermodynamics (GSLT) on the event horizon starting from the first law of thermodynamics. The paper is organized as follows: Effective Friedmann equations in the brane scenario have been formulated in section - II. The validity of GSLT has been examined in sections III and IV for perfect fluid and holographic dark energy respectively. The paper ends with conclusions in section V.

II. RANDALL-SANDRUM -II BRANE MODEL AND EFFECTIVE FRIEDMANN EQUATIONS:

Let us suppose that our n -dimensional visible universe is described by an $(n - 1)$ -brane embedded in an $(n + 1)$ -dimensional AdS space-time (called bulk). Due to string theory effects all ordinary matters are confined in the brane and only gravity can propagate across the brane. At each point on the brane, a space-time unit normal (to the surface) n^A can be defined as $g^{AB}n_A n_B = 1$ where g_{AB} is the bulk metric. The induced metric on the brane is given by

$$q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \quad (1)$$

Here capital letters run over the $(n + 1)$ bulk co-ordinates while the greek indices corresponds to brane-coordinates. The brane can be identified as the hypersurface $\chi = 0$ where χ is a local Gauss normal coordinate such that $n_\mu dx^\mu = d\chi$. If the bulk is assumed to have only a cosmological constant Λ_{n+1} , then the energy-momentum tensor in $(n + 1)$ dimension has the form

$$T_{AB} = -\Lambda_{n+1}g_{AB} + S_{AB}\delta(x) \quad (2)$$

with

$$S_{\mu\nu} = -\lambda q_{\mu\nu} + \tau_{\mu\nu}.$$

Here $\tau_{\mu\nu}$ is the energy-momentum tensor of the matter on $(n - 1)$ brane and λ is the brane tension. The dirac delta function assures the confinement of matter in the brane. Thus considering brane as a singular hyper surface in the bulk we have from the Israel's junction condition , the expression for the extrinsic curvature

$$k_{\mu\nu} = -\frac{1}{2}\kappa_{n+1}^2(S_{\mu\nu} - \frac{1}{(n-1)}q_{\mu\nu}S) \quad (3)$$

where Z_2 symmetry (i.e. reflection symmetry) across the brane has been imposed. Then using the Gauss equation and the bulk Einstein equations the effective Einstein equations on the brane has the form

$$G_{\mu\nu}^{(n)} = -\Lambda_n q_{\mu\nu} + 8\pi G_n T_{\mu\nu} + \kappa_{(n+1)}^4 \Pi_{\mu\nu} \quad (4)$$

where

$$\Lambda_n = \kappa_{n+1}^2 \left[\frac{n-2}{n} \Lambda_{n+1} + \frac{(n-2)}{8(n-1)} \kappa_{n+1}^2 \lambda^2 \right]$$

is the effective cosmological constant on the brane,

$$G_n = \frac{(n-2)\lambda}{32\pi(n-1)} \kappa_{n+1}^4,$$

is the Newton's constant in n -dimensions and

$$\Pi_{\mu\nu} = -\frac{1}{4}T_{\mu\alpha}T_{\nu}^{\alpha} + \frac{1}{4(n-1)}TT_{\mu\nu} + \frac{1}{8}g_{\mu\nu}T_{\alpha\beta}T^{\alpha\beta} - \frac{1}{8(n-1)}T^2q_{\mu\nu}.$$

The above field equations on the brane may be considered as the usual Einstein equations with an effective energy-momentum tensor, given by the right hand side. One may note that the above effective Einstein equations on the brane are not closed as the bulk geometry can not be determined from these equations. Also the standard Einstein equations can be recovered if $\kappa_{n+1} \rightarrow 0$, but G_n is finite. Further, in the low energy limit the quadratic correction term $\Pi_{\mu\nu}$ becomes insignificant and one gets back the usual Einstein equations.

In the following sections, we mainly deal with the dynamical aspects of the brane world cosmology so without any loss of generality we assume the cosmological constant Λ_n (independent of time) to be zero. As a result we have

$$\Lambda_{n+1} = -\frac{n}{8(n-1)}\kappa_{n+1}^2\lambda^2.$$

Let us consider the $(n-1)$ -brane to be n -dimensional FRW universe embedded in a $(n+1)$ -dimensional pure AdS bulk and the perfect fluid is the matter contained in the brane. So the metric on the brane can be written as

$$ds^2 = q_{\mu\nu}dx^{\mu}dx^{\nu} = -dt^2 + \frac{a^2(t)}{1-kr^2}dr^2 + a^2(t)r^2d\Omega_{n-2}^2 \quad (5)$$

$$= h_{ab}dx^adx^b + \tilde{r}^2d\Omega_{n-2}^2$$

where $\tilde{r} = ar$ is the area radius, $x^0 = t$, $x^1 = r$ and $h_{ab} = \text{diag}(-1, \frac{a^2}{(1-kr^2)})$ is the metric of the 2-space.

Using the form of the energy-momentum tensor

$$T_{\mu\nu} = (\rho + p)u_{\mu}u_{\nu} + p q_{\mu\nu} \quad (6)$$

with $u_{\mu}u^{\mu} = -1$, the quadratic correction term $\Pi_{\mu\nu}$ has the expression

$$\Pi_{\mu\nu} = \frac{(n-2)}{8(n-1)}\rho [2(\rho + p)u_{\mu}u_{\nu} + (\rho + 2p)q_{\mu\nu}] \quad (7)$$

Thus the explicit effective Einstein equations on the brane are given by

$$(n-1)(n-2)\left(H^2 + \frac{k}{a^2}\right) = 16\pi G_n \left(\rho + \frac{(n-2)}{(n-1)}\frac{\kappa_{n+1}^4}{64\pi G_n}\rho^2\right) \quad (8)$$

$$-(n-2)\left(\dot{H} - \frac{k}{a^2}\right) = 8\pi G_n \left(\rho + p + \frac{(n-2)}{(n-1)}\frac{\kappa_{n+1}^4}{32\pi G_n}\rho(\rho + p)\right) \quad (9)$$

The energy conservation equation takes the form

$$\dot{\rho} + (n-1)H(\rho + p) = 0 \quad (10)$$

III. STUDY OF GSLT ON THE EVENT HORIZON:

The FRW brane and the effective Einstein equations on it are presented in the previous section. The apparent horizon is characterized by the relation

$$h^{ab}\tilde{r}_{,a}\tilde{r}_{,b} = 0 \quad (11)$$

and we have the radius of the apparent horizon

$$R_A = \frac{1}{\sqrt{(H^2 + \frac{k^2}{a^2})}} \quad (12)$$

where $H = \frac{\dot{a}}{a}$ is the usual Hubble parameter and $k = 0, \pm 1$ for flat, closed and open FRW model.

Assuming the existence of the event horizon (which exists for an accelerated expanding universe in Einstein gravity) the radius of the cosmological event horizon is given by

$$R_E = a \int_t^\infty \frac{dt}{a} = a \int_a^\infty \frac{da}{Ha^2} \quad (13)$$

Thus the apparent horizon, event horizon and the Hubble horizon ($R_H = \frac{1}{H}$) are connected as described in Ref.[6] for flat open and closed model for the universe.

Now we project the effective energy-momentum tensor on the brane in the normal direction to the $(n-1)$ space and define

$$W \equiv -\frac{1}{2}\tilde{T}^{ab}h_{ab} \quad (14)$$

and

$$\psi_a \equiv \tilde{T}_b^a \partial_b \tilde{r} + W \partial_a \tilde{r} \quad (15)$$

where \tilde{T}^{ab} is the projected energy-momentum tensor; W and ψ_a are called the work density and the energy supply vector [7] respectively. The work density W at the horizon may be viewed as the work done by the change of the horizon while the energy-supply at the horizon is total energy flow through the horizon. The area (A) and volume (V) of the $(n-1)$ brane of radius \tilde{r} having explicit form

$$A = (n-1)\Omega_{n-1}\tilde{r}^{n-2} \quad \text{and} \quad V = \Omega_{n-1}\tilde{r}^{n-1} \quad (16)$$

with $\Omega_{n-1} = \frac{\Pi^{\frac{(n-1)}{2}}}{\Gamma(\frac{n-1}{2}+1)}$, volume of an $(n-1)$ D unit ball.

Let us now consider the heat flow δQ through the event horizon for an infinitesimal time δt , assuming the volume of the brane to be unchanged (i.e. $dV = 0$). As this heat flow δQ is equivalent to the amount of energy crossing the event horizon in time δt so we have from clausius relation [8]

$$T_E dS_E = \delta Q = -dE = -A\psi \quad (17)$$

where S_E and T_E are the entropy and temperature on the event horizon.

For the present brane model the explicit expressions of work density and energy supply are given by

$$W = -\frac{1}{2} \left(-\rho + p + \frac{n-2}{4(n-1)} \frac{\kappa_{n+1}^4}{\kappa_n^2} \rho p \right) dt \quad (18)$$

and

$$\psi = -\frac{\kappa_n^2 R_E H}{8\pi G_n}(\rho + p) \left(1 + \frac{n-2}{4(n-1)} \frac{\kappa_{n+1}^4}{\kappa_n^2} \rho \right) dt \quad (19)$$

and we have

$$dS_E = \frac{(n-1)\Omega_{n-1}R_E^{n-1}H\kappa_n^2}{T_E 8\pi G_n}(\rho_{total} + p_{total})dt \quad (20)$$

From Gibb's equation [9]

$$T_E dS_I = dE_I + p dV \quad (21)$$

and we have

$$dS_I = \frac{-(n-1)\Omega_{n-1}R_E^{n-2}(\rho + p)}{T_E} dt \quad (22)$$

where S_I is the entropy of the matter distribution bounded by the event horizon and E_I is the energy of the inside matter with expression

$$E_I = \Omega_{n-1}R_E^{n-1}\rho .$$

It is to be noted that for thermodynamical equilibrium the temperature of the matter distribution is chosen as that of the boundary (the event horizon). Further, in deriving the expression for dS_I from Gibb's equation, we have used the variation of the event horizon

$$\frac{dR_E}{dt} = \left(R_E - \frac{1}{H} \right) H \quad (23)$$

Thus using the energy conservation relation the time variation of the total entropy is given by

$$\frac{d}{dt}(S_E + S_I) = (n-1)\Omega_{n-1}(\rho + p) \frac{R_E^{n-2}H}{T_E} \left[R_E \left(1 + \frac{\rho}{\lambda} \right) - \frac{1}{H} \right] \quad (24)$$

IV. GSLT ON THE EVENT HORIZON OF (N-1)BRANE FILLED WITH HOLOGRAPHIC DARK ENERGY(HDE):

Let us consider the flat FRW model of the universe filled with holographic dark energy [10]. We have the modified Friedmann equation in $(n-1)$ -brane is

$$(n-1)(n-2) \left(H^2 + \frac{k}{a^2} \right) = 16\pi G_n \left(\rho_D + \frac{(n-2)}{(n-1)} \frac{\kappa_{n+1}^4}{64\pi G_n} \rho_D^2 \right) \quad (25)$$

$$-(n-2) \left(\dot{H} - \frac{k}{a^2} \right) = 8\pi G_n \left(\rho_D + p_D + \frac{(n-2)}{(n-1)} \frac{\kappa_{n+1}^4}{32\pi G_n} \rho_D(\rho_D + p_D) \right) \quad (26)$$

where ρ_D and p_D are the energy density and thermodynamic pressure corresponding to HDE. The energy conservation equation takes the form

$$\dot{\rho}_D + (n-1)H(\rho_D + p_D) = 0 \quad (27)$$

where the effective holographic dark energy density [11,12] is given as

$$\rho_D = c^2 (\sqrt{\pi} M_n)^{n-3} M_n A_n^{-1} \frac{n-2}{8\Gamma(\frac{n-1}{2})} R_E^{-2} \quad (28)$$

where $M_n = G_n^{-\frac{1}{n-2}}$, G_n is the n -dimensional gravitational constant; $A_n = \frac{\pi^{\frac{n-1}{2}}}{(\frac{n-1}{2})!}$ and c^2 is a numerical factor.

Now the amount of energy crossing the event horizon during time dt is given by

$$-dE = (n-1)\Omega_{n-1} R_E^{n-1} H(\rho_t + p_t) \quad (29)$$

where ρ_t and p_t is the effective energy density and pressure in brane world scenario with

$$\rho_t + p_t = (\rho_D + p_D) \left(1 + \frac{n-2}{4(n-1)} \frac{\kappa_{n+1}^4}{\kappa_n^2} \rho_D \right) \quad (30)$$

So assuming the validity of the first law of thermodynamics we get

$$dS_E = \frac{(n-1)\Omega_{n-1} R_E^{n-1} H \kappa_n^2}{T_E 8\pi G_n} (\rho_D + p_D) \left(1 + \frac{n-2}{4(n-1)} \frac{\kappa_{n+1}^4}{\kappa_n^2} \rho_D \right) dt \quad (31)$$

Now taking logarithm on both sides of equation (28) and differentiating we get

$$dR_E = \frac{3}{2}(1 + \omega_D) R_E H dt \quad (32)$$

where the equation of state of HDE is $p_D = \omega_D \rho_D$, ω_D is not necessary a constant.

To obtain the variation of the entropy of the fluid inside the event horizon of the brane we use as before the Gibb's equation (21) with

$$E_I = \Omega_{n-1} R_E^{n-1} \rho_D \quad \text{and} \quad V = \Omega_{n-1} R_E^{n-1}$$

and we get

$$\frac{dS_I}{dt} = \frac{(n-1)\Omega_{n-1} R_E^{n-1} H(\rho_D + p_D)}{T_E} \left[\frac{n-1}{2}(1 + \omega_D) - 1 \right] dt \quad (33)$$

Therefore combining (31) and (33) the time variation of total entropy is given by

$$\frac{d}{dt}(S_I + S_E) = \frac{(n-1)\Omega_{n-1} R_E^{n-1} H(\rho_D + p_D)}{T_E} \left[\frac{\rho_D}{\lambda} + \frac{n-1}{2}(1 + \omega_D) \right] \quad (34)$$

where we have used

$$\frac{\kappa_{n+1}^4}{\kappa_n^2} = \frac{4(n-1)}{(n-2)\lambda}$$

which is the relation between the brane tension and the gravitational coupling constants in bulk and brane.

V. CONCLUSIONS:

In the last two sections we have derived expressions for total entropy change with respect to time for perfect fluid and holographic dark energy model respectively. The conclusions for the validity of GSLT are as follows:

For flat ($k = 0$) or open ($k = -1$) FRW model $R_E > \frac{1}{H}$, so if the matter satisfies weak energy condition then GSLT is always valid. However, for closed ($k = +1$) FRW model if $R_E > \frac{1}{H}$ then as before validity of GSLT depends on the weak energy condition. But if for closed model $R_E < \frac{1}{H}$ then also weak energy condition is sufficient for the GSLT to be satisfied provided the expression within the square bracket in equation (24) is positive definite, otherwise GSLT will be valid if the matter does not obey the weak energy condition (i.e. of phantom nature). For validity of GSLT in phantom thermodynamics one may refer to reference [14].

Moreover, if λ is sufficiently small so that the expression within the square bracket is positive for all FRW model (flat, open, closed) then matter should be non-exotic is the only criteria for the validity of the GSLT.

For holographic dark energy model if $\omega_D > -1$ then GSLT is always satisfied i.e. if the holographic dark energy is not exotic then GSLT will be valid. However, for large λ the first term within the square bracket in equation (34) may be insignificant and then GSLT is satisfied without any restriction on the nature of the holographic dark energy.

It is to be noted that for validity of the GSLT we have used the first law of thermodynamics with the assumption that the volume of the brane does not change for an infinitesimal time, but we have no need to use any explicit form of temperature and entropy on the event horizon. However, for equilibrium thermodynamics, we have chosen the temperature of the matter inside the event horizon is the same as the horizon temperature.

If brane tension λ becomes very large then results for both the two cases reduce to those for Einstein Gravity [see Ref. [5] and [13]].

In the literature there are works related to negative brane tension (see ref [15] and there in). Brane model with negative brane tension is commonly known as bouncing brane [16]. Also for $\lambda < 0$ a phantom universe that begins with an accelerated phase can be evolved into a decelerated phase. In the present context if $\lambda < 0$ then validity of GSLT is as follows: For perfect fluid and holographic model there is an upper bound for energy density in quintessence era while for phantom model the energy density has a lower bound (the magnitude of the brane tension) for perfect fluid and GSLT is always valid for HDE model.

The criteria for the validity of GSLT in brane scenario differ insignificantly from that in Einstein gravity. In particular, for small brane tension λ , the condition for validity of GSLT for closed FRW model with perfect fluid as the matter is different from that for Einstein gravity. Also for holographic dark energy model there is no restriction for the validity of GSLT for Einstein gravity while in brane scenario the matter should obey the weak energy condition.

The matter on the brane is assumed to be such that produces accelerated expansion so that event horizon exists.

We shall now make a comparative study of the restrictions for the validity of GSLT that we have obtained in this paper for Brane scenario with those for general relativity and in other gravity theory. First of all for perfect fluid the general restriction is that "GSLT is satisfied for open and flat model provided weak energy condition is satisfied while for closed model, in addition some restrictions among the horizon radii is also needed". This result is true for Einstein gravity [5], $f(R)$ gravity and modified gravity with logarithmic correction in entropy-area relation [17]. However, in the present work in Brane scenario the restriction for GSLT is quite distinct and it depends on the brane tension to a great extent particularly for closed model. For small λ the expression within the square bracket (in eq. (24)) becomes positive so that GSLT can not be satisfied in phantom era. (In scalar tensor theory it is very complicated.[18])

In the second case when universe is filled with holographic dark energy then GSLT is always satisfied both in quintessence and in phantom era without any restriction [13]. This result is true in Einstein gravity [13], $F(R)$ gravity, modified gravity with logarithmic correction in entropy-area relation [17] and also in DGP brane model [19]. But in the present brane model we need restrictions for validity of GSLT. The term containing

brane tension within square bracket (in equation (34)) plays a crucial role for the validity of GSLT. Therefore, we may conclude that the restrictions for the validity of GSLT are quite distinct in brane scenario compared to those in other gravity theories.

For further work, it will be interesting to examine the validity of GSLT for interacting two fluid system and make a comparative study with the corresponding result in general relativity [20] and DGP brane model [21]. Also the first law of thermodynamics on the event horizon will be examined to have any idea about the entropy and temperature in the event horizon.

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